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# Approximation of Expectation of Diffusion Processes based on Lie Algebra and Malliavin Calculus

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In the present paper, we refine the idea in [1] by using notions in [5]. We use the notation in [5] for free Lie algebra. Let $(\Omega,\mathcal{F},P)$ be a probability space and let $\{(B^1(t),\ldots,B^d(t);\ t\in[0,\infty)\}$ be a $d$-dimensional Brownian motion. Let $B^0(t)=t$, $t\in[0,\infty)$. Let $V_0,V_1,\ldots,V_d\in C_b^\infty(\mathbf{R}^N;\mathbf{R}^N)$. Here $C_b^\infty(\mathbf{R}^N;\mathbf{R}^n)$ denotes the space of $\mathbf{R}^n$-valued smooth functions defined in $\mathbf{R}^N$ whose devivatives of any order are bounded. We regard elements in $C_b^\infty(\mathbf{R}^N;\mathbf{R}^N)$ as vector fields on $\mathbf{R}^N$.

Now let $X(t,x)$, $t\in[0,\infty)$, $x\in\mathbf{R}^N$, be the solution to the Stratonovich stochastic integral equation

$$X(t,x)=x+\sum_{i=0}^{d}\int_0^t V_i(X(s,x))\circ dB^i(s). \tag{1}$$

Then there is a unique solution to this equation. Moreover we may assume that with probability one $X(t,x)$ is continuous in $t$ and smooth in $x$.

Let $A=A_d=\{v_0,v_1,\ldots,v_d\}$, be an alphabet, a set of letters, and $A^*$ be the set of words consisting of $A$ including the empty word which is denoted by 1. For $u=u^1\cdots u^k\in A^*$, $u^j\in A$, $j=1,\ldots,k$, $k\geq 0$, we denote by $n_i(u)$, $i=0,\ldots,d$, the cardinal of $\{j\in\{1,\ldots,k\};u^j=v_i\}$. Let $|u|=n_0(u)+\ldots+n_d(u)$, a length of $u$, $\|u\|=|u|+n_0(u)$, and $\#(u)$ denotes the cardinal of $\{i\in\{0,\ldots,d\};n_i(u)\geq 1\}$ for $u\in A^*$. Let $\mathbf{R}\langle A\rangle$ be the $\mathbf{R}$-algebra of noncommutative polynomials on $A$, $\mathbf{R}\langle\langle A\rangle\rangle$ be the $\mathbf{R}$-algebra of noncommutative formal series on $A$, $\mathcal{L}(A)$ be the free Lie algebra over $\mathbf{R}$ on the set $A$, and $\mathcal{L}((A))$ be the $\mathbf{R}$ Lie algebra of free Lie series on the set $A$.

Let $\iota$ denotes the left normed bracketing operator, i.e.,

$$\iota(v_{i_1}\cdots v_{i_n})=[\ldots[v_{i_1},v_{i_2}],\ldots,v_{i_n}].$$

Let $p:\mathbf{R}\langle A\rangle\to\mathbf{R}[x_0,\ldots,x_d]$ denotes a natural homomorphism from the algebra of noncommutative polynomials to the the algebra of commutative polynomials such that $p(u)=x_0^{n_0(u)}\cdots x_d^{n_d(u)}$, $u\in A^*$.

Vector fields $V_0,V_1,\ldots,V_d$ can be regarded as first differential operators over $\mathbf{R}^N$. Let $\mathcal{DO}(\mathbf{R}^N)$ denotes the set of smooth differential operators over $\mathbf{R}^N$. Then $\mathcal{DO}(\mathbf{R}^N)$ is a noncommutative algebra over $\mathbf{R}$. Let $\Phi:\mathbf{R}\langle A\rangle\to\mathcal{DF}(\mathbf{R}^N)$ be a homomorphism given by

$$\Phi(1)=Identity,\qquad \Phi(v_{i_1}\cdots v_{i_n})=V_{i_1}\cdots V_{i_n},\qquad n\geq 1,\ i_1,\ldots,i_n=0,1,\ldots,d.$$

Also, note that

$$\Phi(\iota(v_{i_1}\cdots v_{i_n})) = [\cdots[V_{i_1}, V_{i_2}], \cdots, V_{i_n}], \qquad n \geq 2,\ i_1, \ldots, i_n = 0, 1, \ldots, d.$$

Let $B(t;u)$, $t \in [0,\infty)$, $u \in A^*$, be inductively defined by

$$B(t;1) = 1, B(t;v_i) = B^i(t),\ i = 0, 1, \ldots, d,$$

and

$$B(t;uv_i) = \int_0^t B(s;u) \circ dB^i(s) \qquad u \in A^*,\ i = 0, \ldots, d.$$

Also we define $B(t;w)$ $t \in [0,\infty)$, $w \in \mathbf{R}\langle A\rangle$ by

$$B(t;\sum_{u\in A^*} a_u u) = \sum_{u\in A^*} a_u B(t;u),$$

and we denote $B(1;w)$ by $B(w)$ for $w \in \mathbf{R}\langle A\rangle$.

Let $A^*_m = \{u \in A^*;\ \| u \| = m\}$, $m \geq 0$, and let $\mathbf{R}\langle A\rangle_m = \sum_{u\in A^*_m} \mathbf{R}u$, and $\mathbf{R}\langle A\rangle_{\leq m} = \sum_{k=0}^m \mathbf{R}\langle A\rangle_k$, $m \geq 0$. Let $j_m : \mathbf{R}\langle\langle A\rangle\rangle \to \mathbf{R}\langle A\rangle_{\leq m}$ be a natural sujective linear map such that $j_m(u) = u$, $u \in A^*$, $\| u \| \leq m$, and $j_m(u) = 0$, $u \in A^*$, $\| u \| \geq m+1$. Let $\mathcal{L}(A)_m = \mathcal{L}(A) \cap \mathbf{R}\langle A\rangle_m$, and $\mathcal{L}(A)_{\leq m} = \mathcal{L}(A) \cap \mathbf{R}\langle A\rangle_{\leq m}$, $m \geq 1$. Let $A^{**} = \{u \in A^*;\ u \neq 1, v_0\}$, and $A^{**}_{\leq m} = \{u \in A^{**};\ \| u \| \leq m\}$, $m \geq 1$.

Let $\Psi_s : \mathbf{R}\langle\langle A\rangle\rangle \to \mathbf{R}\langle\langle A\rangle\rangle$, $s > 0$, be given by

$$\Psi_s(\sum_{m=0}^{\infty} x_m) = \sum_{m=0}^{\infty} s^{m/2} x_m, \qquad x_m \in \mathbf{R}\langle A\rangle_m, \quad m \geq 0.$$

Now we introduce a condition (UFG) on the family of vector field $\{V_0, V_1, \ldots, V_d\}$ as follows.

(UFG) There are an integer $\ell$ and $\varphi_{u,u'} \in C_b^\infty(\mathbf{R}^N)$, $u \in A^{**}$, $u' \in A^{**}_{\leq \ell}$, satisfying the following.

$$\Phi(\iota(u)) = \sum_{u'\in A^{**}_{\leq \ell}} \varphi_{u,u'} \Phi(\iota(u')), \qquad u \in A^{**}.$$

Let us define a semi-norm $\| \cdot \|_{V,n}$, $n \geq 1$, on $C_0^\infty(\mathbf{R}^N;\mathbf{R})$ by

$$\| f \|_{V,n} = \sum_{k=1}^{n} \sum_{u_1,\ldots,u_k \in A^{**}, \| u_1\cdots u_k\| = n} \| \Phi(\iota(u_1)\cdots\iota(u_k)) f \|_\infty .$$

Now let us define a semigroup of linear operators $\{P_t\}_{t\in[0,\infty)}$ by

$$(P_t f)(x) = E[f(X(t,x))], \qquad t \in [0,\infty),\ f \in C_b^\infty(\mathbf{R}^N).$$

Then we can prove the following ([2]).

**Theorem 1** *Assume that the family of vector fields satisfies the condition* (UFG). *Then for any $n \geq 1$ there is a constant $C > 0$ such that*

$$\| P_t f \|_{V,n} \leq \frac{C}{t^{n/2}} \| f \|_\infty, \quad f \in C_b^\infty(\mathbf{R}^N),\ t \in (0,1].$$

Let us think of a family $\{Q_{(s)};\ s \in (0,1]\}$ of linear operators in $C_b(\mathbf{R}^N)$.

**Definition 2** *We say that $Q_{(s)}$, $s \in (0,1]$, is m-similar, $m \geq 1$, if there are a constant $C > 0$ and $n \geq m+1$ such that*

$$\| P_s f - Q_{(s)} f(x) \|_\infty \leq C\Big( \sum_{k=m+1}^{n} s^{k/2} \| f \|_{V,k} + s^{(m+1)/2} \| \nabla f \|_\infty\Big),$$

*and*

$$\| Q_{(s)} f - f \|_\infty \leq C s^{1/2} \| \nabla f \|_\infty$$

*for any $s \in (0,1]$, and $f \in C_b^\infty(\mathbf{R}^N;\mathbf{R})$.*

Let $T > 0$ and $\gamma > 0$. Let $t_k = t_k^{(n)} = \dfrac{k^\gamma T}{n^\gamma}$, $n \geq 1$, $k = 0,1,\ldots,n$, and let $s_k = s_k^{(n)} = t_k - t_{k-1}$, $k = 1,\ldots,n$. Then we have the following.

**Theorem 3** *Let $m \geq 1$ and $Q_{(s)}$, $s > 0$ be an m-similar family of linear operators in $C_b(\mathbf{R}^N)$. Then we have the following.*
*For $\gamma \in (0, m-1)$, there is a constant $C > 0$ such that*

$$\| P_T f - Q_{(s_n)} Q_{(s_{n-1})} \cdots Q_{(s_1)} f \|_\infty \leq C n^{-\gamma/2} \| \nabla f \|_\infty, \quad f \in C_b^\infty(\mathbf{R}^N),\ n \geq 1.$$

*For $\gamma = m-1$, there is a constant $C > 0$ such that*

$$\| P_T f - Q_{(s_n)} Q_{(s_{n-1})} \cdots Q_{(s_1)} f \|_\infty \leq C n^{-\frac{m-1}{2}} \log(n+1) \| \nabla f \|_\infty,$$
$$f \in C_b^\infty(\mathbf{R}^N),\ n \geq 1.$$

*For $\gamma > m-1$, there is a constant $C > 0$ such that*

$$\| P_T f - Q_{(s_n)} Q_{(s_{n-1})} \cdots Q_{(s_1)} f \|_\infty \leq C n^{-\frac{m-1}{2}}, \| \nabla f \|_\infty, \qquad f \in C_b^\infty(\mathbf{R}^N),\ n \geq 1.$$

**Definition 4** *We say that a $\mathcal{L}((A))$-valued random variable $Z$ is m-$\mathcal{L}$-moment similar, $m \geq 1$, if*

$$E[\langle j_m(Z), j_m(Z)\rangle^n] < \infty \qquad \text{for any } n \geq 1,$$

*and if*

$$E[j_m(\exp(Z))] = E[j_m(X(1))].$$

**Theorem 5** *Let $m \geq 1$ and $Z$ be a $\mathcal{L}((A))$-valued m-$\mathcal{L}$-moment similar random variable. Also, let $Y : (0,1] \times \Omega \to C(\mathbf{R}^N;\mathbf{R}^N)$ be a measurable map such that*

$$\sup_{s \in (0,1], x \in \mathbf{R}^N} s^{-(m+1)/2} E[|Y(s)(x)|] < \infty$$

*and*

$$E[\sup_{|x| \leq n} |Y(s)(x)|] < \infty, \qquad s \in (0,1],\ n \geq 1.$$

*Let us define a linear map $Q_{(s)}$, $s > 0$, in $C_b(\mathbf{R}^N)$ by*

$$(Q_{(s)} f)(x) = E[f(\exp(\Phi(j_m(\Psi_s(Z))))(x) + Y(s)(x))], \qquad f \in C_b(\mathbf{R}^N).$$

*Then $\{Q_{(s)};\ s \in (0,1]\}$ is m-similar.*

We can show the following characterization theorem for 5-$\mathcal{L}$-moment similar random variables.

**Theorem 6** *Let $Z$ be an $\mathcal{L}_{\le 5}(A)$-valued random variable. Then $Z$ is 5-$\mathcal{L}$ moment similar, if and only if there are random variables $\xi_i$, $i = 1, \dots, d$, $\eta_{ij}$, $1 \le i < j \le d$, $\zeta_{ij}^{(3)}$, $i, j = 1, \dots, d$, $i \ne j$, $\zeta_{0i}^{(4)}$, $i = 1, \dots, d$, and $\mathcal{L}_m(A)$-valed rancom variables $\rho^{(m)}$, $m = 3, 4, 5$, satisfying the following.*

(1) $Z$

$$= \sum_{i=1}^{d} \xi_i v_i + (v_0 + \sum_{1 \le i < j \le d} \eta_{ij}^{(2)} [v_i, v_j]) + ( \sum_{1 \le i \ne j \le d} \zeta_{ij}^{(3)} [[v_i, v_j], v_j] + \rho^{(3)})$$

$$+ (\sum_{i=1}^{d} \zeta_{0i}^{(4)} [[v_0, v_i], v_i] + \rho^{(4)}) + \rho^{(5)}.$$

(2) $E[\xi_i] = E[\xi_i^3] = E[\xi_i^5] = 0$,

$$E[\xi_i^2] = 1, \quad E[\xi_i^4] = 3, \qquad i = 1, \dots, d,$$

$$E[\eta_{ij}] = 0, \quad E[\eta_{ij}^2] = 1, \qquad 1 \le i < j \le d,$$

$$E[\prod_{i=1}^{d} \xi_i^{\alpha_i} \prod_{1 \le i < j \le d} \eta_{ij}^{\beta_{ij}}] = \prod_{i=1}^{d} E[\xi_i^{\alpha_i}] \prod_{1 \le i < j \le d} E[\eta_{ij}^{\beta_{ij}}]$$

*for any non-negative integers $\alpha_i$, $i = 1, \dots, d$, and $\beta_{ij}$, $1 \le i < j \le d$ with $\sum_{i=1}^{d} \alpha_i + \sum_{1 \le i < j \le d} 2\beta_{ij} \le 5$.*

(3) $E[\zeta_{ij}^{(3)}] = 0$, $\quad E[\xi_k \zeta_{ij}^{(3)}] = \frac{1}{12} \delta_{ik}$ *for any* $1 \le i, j, k \le d$, $i \ne j$, *and*

$$E[(\xi_i \xi_j \zeta_{k\ell}^{(3)}] = 0, \qquad 1 \le i, j, k, \ell \le d, \quad k \ne \ell,$$

$$E[(\eta_{ij} \zeta_{k\ell}^{(3)}] = 0 \qquad 1 \le i, j, k, \ell \le d, \quad i < j, \quad k \ne \ell,$$

*and*

$$E[(\prod_{i=1}^{d} \xi_i^{\alpha_i} \prod_{1 \le i < j \le d} \eta_{ij}^{\beta_{ij}}) \rho^{(3)}] = 0$$

*for any non-negative integers $\alpha_i$, $i = 1, \dots, d$, and $\beta_{ij}$, $1 \le i < j \le d$ with $\sum_{i=1}^{d} \alpha_i + \sum_{1 \le i < j \le d} 2\beta_{ij} \le 2$.*

(4) $E[\zeta_{0i}^{(4)}] = \frac{1}{12}$, $\quad E[\xi_j \zeta_{0i}^{(4)}] = 0$, $1 \le i, j \le d$, *and*

$$E[\rho^{(4)}] = E[\xi_i \rho^{(4)}] = 0, \qquad i = 1, \dots, d..$$

(5) $E[\rho^{(5)}] = 0$.

## References

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